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Center-of-Mass Energy Removal in Relativistic Three-Quark Models of the Proton

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A variant of a squared three-body Dirac equation is used to determine center-of-mass energy effects in independent particle motion approximations for three quarks in the nucleon. A scalar linear flux tube potential is used to confine the quarks. The relativistic nearly massless three-quark system, in the rest frame where the total momentum is zero, has a squared energy that is 3/5 the value compared to when the quarks are assumed to move independently. This is smaller than the 2/3 energy ratio determined using the non-relativistic harmonic oscillator model. This analytic model has one parameter, the flux tube constant. Choosing the flux tube constant to reproduce the proton rest energy, results in the analytic wave function well reproducing the proton axial charge and rms charge radius. The proton magnetic moment predicted is 2.235, lower than experiment.

1. INTRODUCTION

In the nonrelativistic three-body problem with harmonic oscillator potentials, one can remove the center-of-mass energy from the Hamiltonian using Jacobi relative coordinates such that the kinetic energy of the center-of-mass motion separates out into a single term (Ballot and Ripelle, 1980; Baz and Zhukov, 1970). Removal of this term from the Schroedinger equation for a system of three equal masses, shows that the system ground state, when the center of mass is fixed, has an energy reduced by 2/3 compared to the energy with independent particle motion. When dealing with small current quark masses (Particle Data Group, 1994), the bound quark problem must be treated relativistically. The one-body Dirac equation for a massless scalar-plus-zeroth-component-of-a-vector quadratic potential has simple analytic solutions (Ferreira *et al.*, 1980; Ferreira and Zagury, 1971, 1977; Smith and Tassie, 1971; Tegen *et al.*, 1982). The three-quark system with independent motion wave functions using this potential has been corrected for the center-of-mass motion (Tegen *et al.*, 1983) by use of a momentum projecting technique. There the composite three-quark wave function was required to have total momentum

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zero. This momentum projection technique did not treat the lower components of the Dirac wave function properly. One can express the three-body Dirac equation kinetic energy in terms of the momentum of the center of mass and the momenta of the relative motion, using Jacobi coordinates (Strobel, 1996), but the operators do not commute relativistically. Provided one stays in the center-of-momentum frame where the total momentum is zero, one can still solve the three-body Dirac equation, with interactions, for system energies without any center-of-mass energy present. This frame is called the rest frame, as the total momentum is zero. This was done for two-body interactions that were an extension of the scalar-plus-vector one-body quadratic potentials. Analytic solutions were obtained (Strobel, 1996) for arbitrary mass, for the case of all three quark masses being the same.

A scalar linear flux-tube potential model (Carlson *et al.*, 1983) inspired from QCD considerations has frequently been used to describe bound quarks. Such a potential, combined with an empirical short-range one-gluon exchange-potential term may describe quark dynamics in the nucleon. After a hyperangular integration is done, the three-body Dirac equation with this potential can be solved by an infinite power series expansion technique, with recurrence relations determining the expansion coefficients (Strobel, 1999). Convergence comes only for the correct guess of the system energy. However, a more nearly analytic approach is sought here.

A squared Hamiltonian approach (Abe and Fujita, 1987) will be used here to get analytic wave functions appropriate for a scalar linear potential. The idea is: Given $H\Psi = E\Psi$, if we square the operator, we obtain $H^2\Psi = E^2\Psi$, and if both are solved exactly, one obtains the same wave function in each case. We seek, and will find, a squared Hamiltonian that we can solve, both with independent particle motion, and again with the center-of-mass motion removed.

With a linear scalar potential, the one-body Dirac equation can be written as

$$[\boldsymbol{\alpha} \cdot p + \boldsymbol{\beta}(m + br)]\psi = E\psi. \tag{1}$$

where m is the quark mass, and b the flux tube constant. $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the Dirac matrices. The quark mass is assumed to be comparable to the current quark masses (Particle Data Group, 1994) as determined by experiments. For the up and down quarks, these masses are taken as very small compared to the proton rest energy. Squaring this one-body equation results in

$$[p^2 + b^2r^2 + m^2 + 2mbr + i\beta\alpha \cdot r]\psi = E^2\psi. \tag{2}$$

If this squared Hamiltonian equation is solved exactly, the solution is the same as solving the Dirac linear Hamiltonian equation. The only nondiagonal term is the $\alpha \cdot r$ term. This is the component of $\boldsymbol{\alpha}$ in the unit radial direction. Fujita and Abe (1987) pioneered the approximation of neglecting this nondiagonal commutator term. With zero mass, the retained terms are just the diagonal harmonic oscillator Hamiltonian. The solution including only the diagonal terms results in uncoupled harmonic oscillator wave functions for the upper and the lower components.

Inclusion of the off diagonal term to first order only, couples the upper and lower components and determines their relative magnitude without changing the shape of the uncoupled wave functions. The resulting calculated energy of such an approach, for massless quarks, is good to three-digit accuracy, as shown in Critchfield's exact numerical solution of the same linear Hamiltonian (Critchfield, 1976). This approximation will be utilized in a three-quark model to estimate the removal of center-of-mass energies in a relativistic three-body Dirac equation. With a quark model of the nucleon, this potential is a reasonable model of a flux-tube potential (Carlson *et al.*, 1983), which minimizes the flux-tube length connecting and confining the three quarks within the nucleon.

We consider the three-body Dirac equation with linear scalar one body potentials as the Hamiltonian for a three-quark model of the nucleon. This work is restricted to a $(1/2^+)^3$ configuration of the three quarks, which is assumed to describe the state of lowest energy for the nucleon. Solving for the third particles energy, and then squaring, resolving for the second particles energy (Moshinsky and Smirnov, 1996) and resquaring twice more results in a harmonic oscillator type equation. The nonrelativistic center-of-mass energy can be removed, by removing the center-of-mass momentum from most terms of the resulting equation. We will consider solely those terms where this removal is possible at first, and then include the rest in a perturbation approach. The resulting wave function will be tested by comparing calculated properties of the proton to experiment, such as its rest energy, magnetic moment, axial charge, and rms charge radius.

2. THEORY

We consider the three-quark Dirac equation Hamiltonian:

$$[\alpha_1 \cdot p_1 + \beta_1(m + br_1) + \alpha_2 \cdot p_2 + \beta_2(m + br_2) + \alpha_3 \cdot p_3 + \beta_3(m + br_3)]\Psi = E\Psi \quad (3)$$

where the subscripts refer to particle label, and Ψ is the composite three-quark wave function. This composite three-quark wave function has eight components. This equation is solved for the Hamiltonian of the third quark, and then squared. Then the result is solved for the Hamiltonian of the second quark, and resquared, as described in Moshinsky and Smirnov's book (Moshinsky and Smirnov, 1996). The resulting 4th order equation for the energy squared can be written as

$$[(E^2 - W^2)^4 + V]\Psi = 0 \quad (4)$$

where

$$\Pi_s^2 = p_s^2 + b^2 r_s^2 + m^2 + 2br_s + i\beta_s\alpha_s \cdot r_s. \quad (5)$$

Here s is 1, 2, or 3 and again refers to particle label. W^2 is defined as

$$W^2 = \Pi_1^2 + \Pi_2^2 + \Pi_3^2. \tag{6}$$

Neglecting the off-diagonal terms in W^2 results in an uncoupled three-body harmonic oscillator system from which the center-of-mass energy can be removed. Further, each of the eight components of this uncoupled three-body system can be solved exactly in a hyperspherical approach, resulting in a single configuration describing the system. The remainder of the equation defining Ψ, from which the nonrelativistic center-of-mass energy cannot easily be removed, is called V. It is

$$\begin{aligned} \mathrm{V} = &-8(E^4 + W^4)\left(\Pi_1^2\Pi_2^2 + \Pi_1^2\Pi_3^2 + \Pi_2^2\Pi_3^2\right) - 16E^2\left[\Pi_1^2\Pi_2^2\Pi_3^2\right. \\ &\left. - \Pi_1^4\left(\Pi_2^2 + \Pi_3^2\right) - \Pi_2^4\left(\Pi_1^2 + \Pi_3^2\right) - \Pi_3^4\left(\Pi_1^2 + \Pi_2^2\right)\right] \\ &+ 16\left(\Pi_1^2\Pi_2^2 + \Pi_1^2\Pi_3^2 + \Pi_2^2\Pi_3^2\right)^2. \end{aligned} \tag{7}$$

The plan is to solve Eq. (4) using perturbation theory, first neglecting the term V. One then can solve the unperturbed equation, after extracting the fourth root as

$$[E^2 - W^2]\Psi = 0. \tag{8}$$

With the solution to the unperturbed equation found, the perturbation theory is used to solve Eq. (4) to determine the energy of the perturbed system. This is done twice, once with independent particle motion, and then with the center-of-mass energy removed.

3. SOLUTION WITH INDEPENDENT PARTICLE MOTION

Consider the $(1/2^+)$ one-body state, where the upper component has orbital angular momentum $L = 0$ and the lower component has $L = 1$. The uncoupled eigenfunctions are separately normalized harmonic oscillator wave functions. Including the off diagonal term to Eq. (2), and writing the wave function as a column vector,

$$\begin{aligned} \psi = &|A_1 R_0(r)| \\ &|B_1 R_1(r)| \end{aligned} \tag{9}$$

where A_1, B_1 are determined from solving the determinant of the matrix for the square of the one-body energy, E_{1b}:

$$\begin{aligned} &\left\lceil \left(3b - E_{1b}^2\right) - nb \right\rceil |A_1| \\ &\left\lfloor -nb\left(5b - E_{1b}^2\right) \right\rfloor |B_1| = 0 \end{aligned} \tag{10}$$

where $n^2 = 8/3\pi$. This comes from the overlap of the uncoupled upper and lower radial wave functions. The relative normalization is that $A_1^2 + B_1^2$ sum to unity.

Including the $2mbr$ term as a perturbation for small m, and expressing m in units of $b^{1/2}$, one obtains

$$\left(E_{1b}^2 - m^2 b\right)/b = 4 + [14m/3\pi^{1/2}] - [1 + (8/3\pi) + (4m/3\pi^{1/2}) + 4m^2/\pi]^{1/2}. \quad (11)$$

For the massless case, this one-body squared energy is $2.64117b$. Now we solve the squared Hamiltonian, Eq. (8), for the case of independent motion of three particles. Since the particles move independently, the solution for the three-body system energy squared, called W_0^2, as V is neglected is

$$W_0^2 = 3\,E_{1b}^2. \quad (12)$$

The composite three-quark wave function has eight components, the relative amplitude of which can be found from expanding out the expression $(A_1 + B_1)^3$. The upper, upper, upper component of the composite wave function that survives in the nonrelativistic limit has a relative amplitude of A_1^3. The lower cubed component of the wave function has a relative amplitude of B_1^3, etc. A_1 and B_1 are independent of the flux-tube constant b.

The quartic equation for E^2 including V is now solved with independent particle motion, using the same approximations leading to Eqs. (9) and (10). Namely, the off diagonal terms are treated to first order only, and the composite three-quark wave function is a product of harmonic oscillator wave functions. Including the potential V, Eq. (8) has four solutions for the energy squared: a triple root, $E_{\mathrm{ip}}^2 = 3E_{1b}^2$, and a fourth solution $E_{\mathrm{ip}}^2 = 9E_{1b}^2$. With the assumption of independent particle motion, the wave function including the V term is the same as the wave function omitting the V term. The energy solution of interest is the fourth, for which we can take the square root, and obtain for independent particle motion, $E_{\mathrm{ip}} = 3E_{1b}$.

4. SOLUTION IN CENTER-OF-MOMENTUM FRAME

Neglecting the perturbation, V, and the nondiagonal terms in W^2, for massless quarks, the quartic equation for the energy squared uncouples. In the center-of-momentum frame, the total momentum is zero and the center-of-mass energy is removed from the kinetic energy terms. Hyperspherical coordinates are used and the three quadratic potential energy terms in W^2 combine as

$$b^2\left(r_1^2 + r_2^2 + r_3^2\right) = b^2\rho^2. \quad (13)$$

where the hyperradius is ρ. We express each of the eight components of the composite three-quark wave function in hyperspherical coordinates as $A_k U_k(\Omega) R_k(\rho)/\rho^{5/2}$. The total orbital angular momentum in each component is denoted by k. A_k is a temporarily unknown relative normalization coupling constant. The second factor accounts for the hyperangular dependence (Baz and Zhukov, 1970), the third is the generally unknown hyperradial dependence of the wave function component. For the uncoupled massless case, it is a simple harmonic oscillator wave function as it satisfies

$$[-d^2/d\rho^2 + b^2\rho^2 + \Lambda(\Lambda+1)/\rho^2]R_k = U^2 R_k. \tag{14}$$

where $\Lambda = k + 3/2$ and $k = L_1 + L_2 + L_3$ is the sum of the orbital angular momenta in each component. The uncoupled eigenvalue $U^2 = b\,(2k+6)$. The norm of the uncoupled wave functions is

$$\int d\rho\, R_k^2(\rho) = 1. \tag{15}$$

The hyperradial part of the uncoupled wave function is

$$R_k(\rho) = N_k\, \rho^{k+5/2} \exp(-b\rho^2/2) \tag{16}$$

where the uncoupled normalization determines

$$N_k^2 = 2b^{k+3}/(k+2)! \tag{17}$$

The eight various components have k values that range from 0 to 3. Including the non diagonal terms of W^2 couples the previously uncoupled components. Symmetries (Strobel, 1996) allow the transposed wave function to be written as a row vector:

$$\psi^t = [A_0R_0,\, A_1R_1,\, A_1R_1,\, A_2R_2,\, A_1R_1,\, A_2R_2,\, A_2R_2,\, A_3R_3]. \tag{18}$$

Including the off diagonal elements of W^2 to first order allows the relative amplitudes of the components A_0, A_1, A_2, and A_3 to be determined by setting to zero the determinant:

$$\begin{vmatrix} (6-X) & -\sqrt{3} & 0 & 0 \\ -\sqrt{3} & (8-X) & -1 & 0 \\ 0 & -4 & (10-X) & -1/\sqrt{5} \\ 0 & 0 & -3\sqrt{5} & (12-X) \end{vmatrix} \begin{vmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{vmatrix} = 0 \tag{19}$$

Here X is the squared eigen-energy over the string constant b. The smallest root is sought, and one finds $X = 4.7573$ approximately. Comparing Eqs. (10) and (19), the nonrelativistic ratio of 2/3 for the energy without and with independent particle motion is seen in the 6 of the first element of (19) and three times the first element of (10). If only the components surviving in the nonrelativistic limit mattered, that 6–9 ratio of the energy squared would be recovered here. However, the lower components of the wave function matter with small quark masses. When the center of mass is constrained to not move, the lower components have a larger contribution to the normalization, than in the independent particle motion case. Including the smaller components, we find $X/W_0^2 = 4.7573/7.9235 = 0.6004$ or about 3/5.

The rest frame eigenvalue of Eq. (4) is now obtained in a perturbative approach. The perturbation V is evaluated by replacing Π_s^2 by y equals cE_{1b}^2. Thus the operator is replaced by its eigenvalue assuming independent particle motion, but now reduced by a factor c, taken as 3/5. This reduction is associated with center-of-mass motion and justified by being determined analytically when V was neglected. With $W_0^2 = 3cE_{1b}^2$, then including V in the fourth order equation for the energy squared, one finds

$$[E^8 - 12yE^6 + 30y^2E^4 - 28y^3E^2 + 9y^4]\Psi = 0 \tag{20}$$

This has the solutions $E^2 = cE_{1b}^2$, a triple root, and the desired root $E^2 = 9cE_{1b}^2$. Thus the energy with the center-of-mass energy approximately removed is found as $E = [3/5]^{1/2}E_{\mathrm{ip}}$.

5. NUCLEON PROPERTIES IN THE SQUARED FLUX TUBE HAMILTONIAN APPROACH

The contribution to the normalization, of the eight components of the wave function are, with independent particle motion, $S = 0.65389$, $P = 0.2984$, $D = 0.04539$, and $F = 0.0023$, while in the rest frame, $S = 0.59789$, $P = 0.30776$, $D = 0.08885$, and $F = 0.00549$. Here S, P, D, and F, are labels denoting 0, 1, 2, or 3 units of orbital angular momentum in the $(1/2^+)^3$ configuration composite wave function component. The upper component contributes 0.86795 to the norm with independent particle motion. In the rest frame, the upper component contributes 0.83268 to the normalization. The removal of the center-of-mass motion thus increases the lower component contribution to the norm.

The axial charge calculated is Ga = 1.373, with independent particle motion, and Ga = 1.2969 in the rest frame. Both values are reasonable, but the rest frame provides closer agreement with the experimental value of 1.26 (Particle Data Group, 1994). Other properties of the proton within the rest frame can be calculated, given a value for the flux tube constant. The proton rest energy is reproduced if the flux tube constant b, is taken as 0.061716 GeV^2. With this value for the flux tube constant, the rms charge radius is then calculated as 0.8339 fermi, in good agreement with experiment. The magnetic moment calculated is 2.235 nuclear magnetons. This is less than experiment (2.793). An anomalous quark magnetic moment can be invoked to explain the difference (Strobel, 1998).

6. SUMMARY

A scalar linear potential is used to bind three quarks within the nucleon. The Dirac Hamiltonian is squared, resquared, and again resquared resulting in a fourth order equation for the energy squared. This problem is solved for independent particle motion, and for motion in the rest frame where the total momentum of the system is zero. Including only diagonal terms to the squared Hamiltonian, one obtains uncoupled harmonic oscillator solutions. Including the off diagonal terms to lowest order couples the upper and lower component with definite coefficients. The energy squared of the three-particle system is reduced by 3/5 when the center-of-mass motion is removed. This difference from 2/3 comes from the inclusion of the lower components of the composite three quark wave function. The rest frame energy of the three quark system is then estimated as 3 $(3/5)^{1/2} E_{1b}$. E_{1b} is the one-body Dirac equation energy. This rest frame energy is larger than the 2/3 energy ratio determined using the nonrelativistic harmonic oscillator model. This analytic model has one parameter, the flux tube constant. Choosing the flux tube constant to reproduce the proton rest energy, allows the analytic wave function to well reproduce the proton axial charge and rms charge radius. The proton magnetic moment predicted is 2.235, lower than experiment.

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